3. Chibrikova, L. I., On boundary value problems of a rectangle. Boundary Value Problems of Functions of a Complex Variable. Kazansk. Univ., Kazan', 1964.
4. Akhiezer, N. I., Elements of the Theory of Elliptic Functions. "Nauka", Moscow, 1971.
5. Dykhne, A. M., Anomalous resistance of plasma in a strong magnetic field. ZhETF, Vol. 59, №2(8), 1970.

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EXPLOSION IN A VARIABLE-DENSITY MEDIUM IN THE PRESENCE OF VARIABLE COUNTERPRESSURE

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The solution of the non-self-similar problem of explosion in a medium with variable initial density whose distribution is subject to power law is considered with variable initial pressure taken into consideration. An exact analytical solution is obtained in particular cases for the initial phase of explosion. The dependence of dimensionless parameters of motion on the geometric coordinate and the shock wave radius, which is obtained by solving differential equations, is derived in the solution of the complete non-self-similar problem. Derived solutions are used for calculating cases of spherical and cylindrical symmetry of explosion for various values of the determining parameters.

The one-dimensional self-similar problem of a strong point explosion was formulated and solved by Sedov [1, 2] on the assumption that the initial pressure of gas, which is small in comparison with the pressure at the front, can be neglected and that the initial density is constant. Strong explosion in a medium of varying density dependent on the geometric coordinate according to the power law was considered in [1, 3]. When counterpressure is taken into consideration, the problem becomes non-self-similar. Its numerical solution appeared in several publications [4 – 9], in which initial pressure was assumed constant.

The non-self-similar problem of explosion in a medium of varying initial density ρ_1 and varying initial pressure p_1 is considered here. These parameters are defined by

$$\rho_1 = Ar^{-\omega}, \qquad p_1 = Cr^{-\varkappa} \tag{0.1}$$

If $\varkappa = 2\omega - 2$, then, in the presence of a gravitational field, the initial density and pressure distributions (0.1) satisfy the equilibrium equations of the medium [1]. A particular case of this problem in linearized formulation for $\varkappa = \omega$ was investigated in [12, 13].

Considerable calculation difficulties encountered in non-self-similar problems have led to the appearance of several approximate methods [3–7, 11]. Sedov had suggested to construct approximate solutions of problems of unsteady motion of gas within a shock wave by using interpolation formulas for the basic functions defining the motion of gas [1].

It is shown here that in investigations of explosion in a gas of varying density and initial pressure (0.1) interpolation formulas can only by applied to one function of gas motion. The remaining values are determined by exact equations of motion. The coefficients in interpolation formulas can be determined by the general and boundary conditions of the problem, as is done in approximate theories of the boundary layer.

1. Statement of problem and fundamental equations. In gasdynamics one-dimensional perturbed motion of a medium in an explosion is defined by a system of equations of the form [2]

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0, \qquad \frac{\partial \rho}{\partial t}+\frac{\partial \rho v}{\partial r}+\frac{(\nu-1)\rho v}{r}=0$$
$$\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial r}+\gamma p\left(\frac{\partial v}{\partial r}+\frac{\nu-1}{r}v\right)=0 \tag{1.1}$$

where t is the time, r is the Euler coordinate, v is the velocity, p is the pressure, ρ is the density, γ is the adiabatic exponent, and $\nu=1, 2, 3$ in the case of plane, cylindrical, and spherical symmetry, respectively. The problem of explosion requires that the solution of the system of Eqs. (1.1) satisfies at the shock wave front $(r=r_2)$ three boundary conditions

$$v(r_2, t)=v_2, \quad \rho(r_2, t)=\rho_2, \quad p(r, t)=p_2 \tag{1.2}$$

where

$$v_2=\frac{2c}{\gamma+1}\left[1-\frac{a^2}{c^2}\right], \qquad \rho_2=\frac{\gamma+1}{\gamma-1}\rho_1\left[1+\frac{2}{\gamma-1}\frac{a^2}{c^2}\right]^{-1}$$
$$p_2=\frac{2\rho_1c^2}{\gamma+1}\left[1-\frac{\gamma-1}{2\gamma}\frac{a^2}{c^2}\right], \qquad c=\frac{dr_2}{dt}, \qquad a^2=\frac{\gamma p_1}{\rho_1} \tag{1.3}$$

Functions $v_2(t)$, $\rho_2(t)$ and $p_2(t)$ are a priori unknown, and their determination is tantamount to the determination of the shock wave radius $r_2(t)$. At the center of symmetry we have in addition to conditions (1.2) the boundary condition for velocity

$$v(0, t)=0 \tag{1.4}$$

At the instant $t=0$ the finite energy E_0 is released and the following initial conditions are specified:

$$v(r, 0)=0 \qquad \rho(r, 0)=Ar^{-\omega}$$
$$p(r, 0)=Cr^{-\varkappa}, \qquad r_2(0)=0 \tag{1.5}$$

The system of determining parameters shows that, if the dimensionless values

$$f=\frac{v}{c}, \qquad g=\frac{\rho}{\rho_2}, \qquad h=\frac{p}{p_2} \tag{1.6}$$

are taken as the unknown functions, these will depend on two dimensionless variables for which we select

$$\lambda=\frac{r}{r_2}, \qquad q=\frac{a^2{}_1}{c^2}, \quad a_1=a(r_2) \tag{1.7}$$

and on constant parameters ν, γ, ω and $\varkappa$.

2. Solution of the problem in a linearized formulation. In the initial phase of an explosion, when the explosion wave is still fairly strong, the variable q is small and the solution of the stated problem can be derived by the method of linearization with reference to the known self-similar solution. In dimensionless variables (1.6) and (1.7) system (1.1) is of the form

$$(f-\lambda)\frac{\partial f}{\partial\lambda}+\frac{(\gamma-1+2q)[2\gamma-(\gamma-1)q]}{\gamma(\gamma+1)^2}\frac{1}{g}\frac{\partial h}{\partial\lambda}+ \\ +\left(\frac{\partial f}{\partial q}-\frac{f}{2q}\right)r_2\frac{dq}{dr_2}+\frac{(\omega+\varkappa)}{2}f=0 \qquad (2.1)$$

$$(f-\lambda)\frac{\partial\ln g}{\partial\lambda}+\frac{\partial f}{\partial\lambda}+\frac{(\nu-1)f}{\lambda}+\left(\frac{\partial\ln g}{\partial q}-\frac{2}{\gamma-1+2q}\right)r_2\frac{dq}{dr_2}-\omega=0$$

$$(f-\lambda)\frac{\partial\ln h}{\partial\lambda}+\gamma\left(\frac{\partial f}{\partial\lambda}+\frac{(\nu-1)}{\lambda}f\right)+\left(\frac{\partial\ln h}{\partial q}-\right. \\ \left.-\frac{2\gamma}{[2\gamma-(\gamma-1)q]q}\right)r_2\frac{dq}{dr_2}-\varkappa=0$$

We introduce dimensionless radius and time and, also, the dimensionless radius of the shock wave

$$R=\frac{r}{r^\circ},\qquad \tau=\frac{t}{t^\circ},\qquad R_2(q)=\frac{r_2}{r^\circ}$$

$$r^\circ=\left(\frac{E_0}{c}\right)^n,\quad t^\circ=r^{\circ m}\left(\frac{A}{\gamma c}\right)^{1/2},\quad n=\frac{1}{\nu-\varkappa},\quad m=1-\frac{\omega-\varkappa}{2} \qquad (2.2)$$

To obtain the complete solution of the explosion problem it is necessary to determine functions $R_2(q)$, $f(\lambda, q)$, $g(\lambda, q)$ and $h(\lambda, q)$ inside the square $0\leqslant\lambda\leqslant 1$ and $0\leqslant q\leqslant 1$ in the plane λ, q. These functions must satisfy the following boundary and initial conditions:

$$f(1,q)=\frac{2}{\gamma+1}(1-q),\quad g(1,q)=h(1,q)=1,\quad f(0,q)=0 \\ f(\lambda,0)=f_0(\lambda),\quad g(\lambda,0)=g_0(\lambda),\quad h(\lambda,0)=h_0(\lambda) \qquad (2.3)$$

where $f_0(\lambda)$, $g_0(\lambda)$ and $h_0(\lambda)$ are known functions which for $q=0$ correspond to the self-similar problem [2].

For $\omega<\nu$ the mass of gas in the spherical volume which contains the coordinate origin, and the rate of shock wave propagation is at the initial stage finite [1]. If $\omega<\nu$ and $\nu-\varkappa>0$, then at the initial explosion stage q is small, hence we can seek the linearized solution of the form [3]

$$f(\lambda,q)=f_0(\lambda)+qf_1(\lambda)+\cdots \\ g(\lambda,q)=g_0(\lambda)+qg_1(\lambda)+\cdots \\ h(\lambda,q)=h_0(\lambda)+qh_1(\lambda)+\cdots \\ \frac{dq}{dR_2}=\frac{q}{nR_2(1+A_1q+\cdots)} \qquad (2.4)$$

Substituting in the system of Eqs. (2.1) the expressions for $f(\lambda, q)$, $g(\lambda, q)$ and $h(\lambda, q)$ from (2.4) and neglecting terms of the order q^2 and higher, we obtain for the determination of functions $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$, and constant A_1 a system of linear differential equations of the form

$$(f_0-\lambda)g_0f_1'+\frac{2(\gamma-1)}{(\gamma+1)^2}h_1'+\left[f_0'+\left(\frac{\omega-\varkappa}{2}+\frac{1}{2n}\right)\right]g_0f_1+ \\ +\left[(f_0-\lambda)f_0'+\left(\frac{\omega-\varkappa}{2}-\frac{1}{2n}\right)f_0\right]g_1+\frac{4\gamma-(\gamma-1)^2}{\gamma(\gamma+1)^2}h_0'+\frac{1}{2n}f_0g_0A_1=0 \qquad (2.5)$$

$$g_0 f_1' + (f_0 - \lambda) g_1' + \left(\frac{\nu - 1}{\lambda} g_0 + g_0'\right) f_1 + \qquad \text{(Cont.)}$$
$$+ \left(\frac{\nu - 1}{\lambda} f_0 + f_0' + \frac{1}{n} - \omega\right) g_1 - \frac{2}{(\gamma - 1) n} g_0 = 0$$
$$\gamma h_0 f_1' + (f_0 - \lambda) h_1' + \left(h_0' + \gamma \frac{\nu - 1}{\lambda} h_0\right) f_1 + \gamma \left(f_0' + \frac{\nu - 1}{\lambda} f_0\right) h_1 +$$
$$+ \frac{1}{n}\left(A_1 - \frac{\gamma - 1}{2\gamma}\right) h_0 - \varkappa h_1 = 0$$

Let us transform system (2.5) to a form more convenient for subsequent analysis. For this we introduce the unknown functions $F(\lambda)$, $G(\lambda)$ and $H(\lambda)$ which we relate to $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$ by formulas

$$f_1(\lambda) = (f_0 - \lambda) F(\lambda), \qquad g_1(\lambda) = g_0 G(\lambda), \qquad h_1(\lambda) = h_0 H(\lambda) \qquad (2.6)$$

After transformation of (2.6), we obtain the system of equations in the new unknown formulas

$$(f_0 - \lambda)^2 F' + \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{h_0}{g_0} H' + (f_0 - \lambda)\left[2f_0' - 1 + \frac{\omega - \varkappa}{2} + \frac{1}{2n}\right] F +$$
$$+ \frac{2(\gamma - 1) h_0'}{(\gamma + 1)^2 g_0} H + \left[(f_0 - \lambda) f_0' + \frac{\omega - \varkappa}{2} - \frac{1}{2n}\right] G +$$
$$+ \frac{4\gamma - (\gamma - 1)^2 h_0'}{\gamma(\gamma + 1)^2 g_0} + \frac{1}{2n} f_0 A_1 = 0$$
$$(f_0 - \lambda) F' + (f_0 - \lambda) G' + (f'_0 - 1) F + \left(\frac{\nu - 1}{\lambda} + \frac{g_0'}{g_0}\right)(f_0 - \lambda) F + \qquad (2.7)$$
$$+ \left(\frac{\nu - 1}{\lambda} f_0 + f_0' + \frac{1}{n} - \omega\right) G + (f_0 - \lambda) \frac{g_0'}{g_0} G - \frac{2}{(\gamma - 1) n} = 0$$
$$\gamma (f_0 - \lambda) F' + (f_0 - \lambda) H' + (f_0 - 1) \gamma F + (f_0 - \lambda)\left[\frac{h_0'}{h} + \gamma \frac{\nu - 1}{\lambda}\right] F +$$
$$+ (f_0 - \lambda) \frac{h_0'}{h_0} H + \gamma \left[f_0' + \frac{\nu - 1}{\lambda} f_0 - \varkappa\right] + \frac{1}{n}\left(A_1 - \frac{\gamma - 1}{2\gamma}\right) = 0$$

with boundary conditions

$$F(1) = \frac{2}{\gamma - 1}, \qquad G(1) = 0, \qquad H(1) = 0 \qquad (2.8)$$

In the general case (arbitrary ω) the solution of the linearized problem of explosion reduces to numerical integration of the system of Eqs. (2.7) with boundary conditions (2.8). The coefficients of this system are determined by the self-similar solution of the problem. If, however,

$$\omega = \omega_1 = \frac{3\nu - 2 + \gamma(2 - \nu)}{\gamma + 1} \qquad (2.9)$$

a closed solution can be obtained, since for this value of ω the self-similar solution has the simple form

$$f_0(\lambda) = \frac{2}{\gamma + 1}\lambda, \qquad g_0(\lambda) = \lambda^{\nu - 2}, \qquad h_0(\lambda) = \lambda^{\nu} \qquad (2.10)$$

Substituting the expressions for $f_0(\lambda)$, $g_0(\lambda)$ and $h_0(\lambda)$ from (2.10) for the coefficients in Eqs. (2.7), we obtain a system of three ordinary inhomogeneous equations with coefficients dependent on parameters γ and ν

$$\lambda F' + \frac{2}{\gamma - 1}\lambda H' + \frac{2\nu}{\gamma - 1} H - \frac{1}{2(\gamma - 1)}\left[8 + \left(\frac{1}{n} + 2b_3 - 2\right)(\gamma + 1)\right] F +$$
$$+ \frac{1}{(\gamma - 1)^2}\left[2(1 - \gamma) - \left(\frac{1}{n} - 2b_3\right)(\gamma + 1)\right] G +$$

$$+\frac{\nu}{(\gamma-1)^2}\left[\frac{4\gamma-(\gamma-1)^2}{\gamma}+\frac{\gamma+1}{\nu n}A_1\right]=0 \tag{2.11}$$
$$\lambda F'+\lambda G'+2(\nu-1)F-\frac{\gamma+1}{\gamma-1}2(2\nu+\gamma-1)\left[\frac{1}{n}+b_2-\nu\right]G+\frac{2(\gamma+1)}{(\gamma-1)^2 n}=0$$
$$\gamma\lambda F'+\lambda H'+\nu(\gamma+1)F-\frac{\gamma+1}{\gamma-1}(\nu+\gamma b_1)H+\frac{\gamma+1}{(\gamma-1)n}\left(\frac{\gamma-1}{2\gamma}-A_1\right)=0$$

where $b_3=(\omega-\varkappa)/2$, $b_2=-\omega$ and $b_1=-\varkappa$. If $\ln\lambda$ is taken as the independent variablè, (2.11) becomes a system of equations with constant coefficients. In the general case the characteristic equation of system (2.11) for $\gamma=3$ and $\nu=2,3$ is cubic, while for $\nu=1$ and any γ it separates into a linear and a quadratic equation. Having derived the solution of system (2.11), we revert to the original unknown functions with the use of formulas (2.6) and obtain

$$f_1(\lambda)=\frac{1-\gamma}{1+\gamma}\lambda\left[\alpha_1+C_1\lambda^{k_1}+C_2\lambda^{k_2}+C_3\lambda^{k_3}\right]$$
$$g_1(\lambda)=\lambda^{\nu-2}\Big[\alpha_2+\frac{(k_1+2\nu-2)(\gamma-1)}{\nu(\gamma+1)-k_1(\gamma-1)+2(\nu-1)(\gamma-3)}C_1\lambda^{k_1}+$$
$$+\frac{(k_2+2\nu-2)(\gamma-1)}{\nu(\gamma+1)-k_2(\gamma-1)+2(\nu-1)(\gamma-3)}C_2\lambda^{k_2}+$$
$$+\frac{(k_3+2\nu-2)(\gamma-1)}{\nu(\gamma+1)-k_3(\gamma-1)+2(\nu-1)(\gamma-3)}C_3\lambda^{k_3}\Big]$$
$$h_1(\lambda)=\lambda^{\nu}\Big[\alpha_3+\frac{[k_1\gamma+\nu(\gamma+1)](\gamma-1)}{\nu(\gamma+1)-k_1(\gamma-1)+2(\nu-1)\gamma(\gamma-3)}C_1\lambda^{k_1}+ \tag{2.12}$$
$$+\frac{[k_2\gamma+\nu(\gamma+1)](\gamma-1)}{\nu(\gamma+1)-k_2(\gamma-1)+2(\nu-1)\gamma(\gamma-3)}C_2\lambda^{k_2}+$$
$$+\frac{[k_3\gamma+\nu(\gamma+1)](\gamma-1)}{\nu(\gamma+1)-k_3(\gamma-1)+2(\nu-1)\gamma(\gamma-3)}C_3\lambda^{k_3}\Big]$$

where C_1, C_2 and C_3 are arbitrary constants, k_1, k_2 and k_3 are the roots of the characteristic equation of the system (2.11), and

$$\alpha_i=B_i+A_1D_i, \qquad i=1,2,3$$
$$B=\nu+\gamma b_1,\quad D=2(2\nu+\gamma-1)+(b_2+1/n-\nu)(\gamma+1)$$
$$B_1=\frac{2\gamma BB_3-(\gamma-1)/n}{2\nu\gamma(\gamma-1)},\quad D_1=\frac{BD_3+1/n}{\nu(\gamma-1)},\quad D_2=\frac{2(\nu-1)[BD_3+1/n]}{\nu(\gamma-1)D}$$
$$B_2=\frac{1}{\nu\gamma(\gamma-1)D}\left\{2\gamma(\nu-1)BB_3+\frac{1}{n}[2\nu\gamma(\gamma+1)-(\nu-1)(\gamma-1)]\right\}$$
$$B_3=\frac{\nu^2}{\gamma\Delta}[(\gamma-1)^2-4\gamma]D(\gamma-1)-\frac{2}{\gamma\Delta}\left[1-\gamma-(\gamma+1)\left(\frac{1}{2n}-b_3\right)\right]\times$$
$$\times\left[\frac{2}{n}(\gamma+1)\nu\gamma-\frac{(\nu-1)(\gamma-1)}{n}\right]-\frac{(\gamma-1)^2}{2\gamma\Delta}\left[4+(\gamma+1)\left(\frac{1}{2n}+b_3-1\right)\right]\frac{D}{n}$$
$$D_3=\frac{\gamma-1}{\nu\Delta}\left[4+(\gamma+1)\left(\frac{1}{2n}+b_3-1\right)\right]\frac{D}{n}-\frac{\gamma^2-1}{\Delta}\frac{D}{n}-$$
$$-\frac{4(\nu-1)}{\nu\Delta}\left[1-\gamma-(\gamma+1)\left(\frac{1}{2n}-b_3\right)\right]\frac{1}{n}$$
$$\Delta=2\nu^2(\gamma-1)^2D-(\gamma-1)\left[4+(\gamma+1)\left(\frac{1}{2n}-1+b_3\right)\right]BD+$$
$$+4(\nu-1)\left[1-\gamma-(\gamma+1)\left(\frac{1}{2n}-b_3\right)\right]B$$

Constants C_1, C_2, C_3 and A_1 are determined by the boundary conditions for functions $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$. Having determined A_1, we find $R_2(q)$ and $\tau(q)$. For a self-similar solution the following relationships are valid:

$$R_2(q) = A_0 q^n, \quad \tau(q) = \delta A_0{}^m q^{n/q}, \quad A_0 = \frac{\delta^{2n}}{(\gamma\alpha_0)^n}, \quad \delta = \frac{2}{\nu + 2 - \omega} \tag{2.13}$$

Taking into consideration (2.13), from (2.4) for the linearized problem we obtain

$$R_2(q) = A_0 q^n \exp(nA_1 q), \quad \tau(q) = \delta A_0{}^m q^{n/\delta}\left[1 + \frac{mn}{n+\delta} A_1 \delta\right] \tag{2.14}$$

Curves of functions $f_1(\lambda)$, $g_1(\lambda)$ and $h_1(\lambda)$ determined by formulas (2.12) and boundary conditions are shown in Figs. 1–3 for several ν, $\varkappa$ and γ.

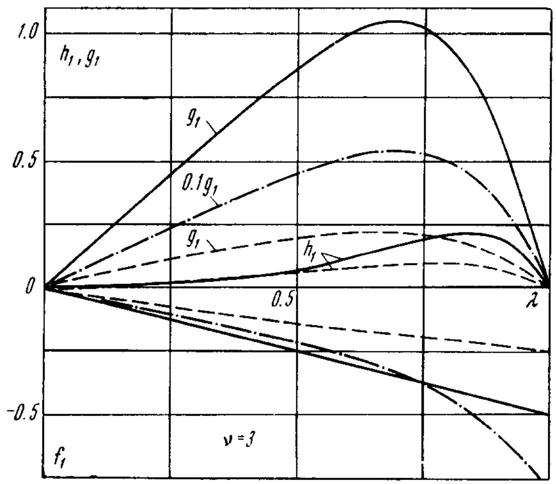

Fig. 1

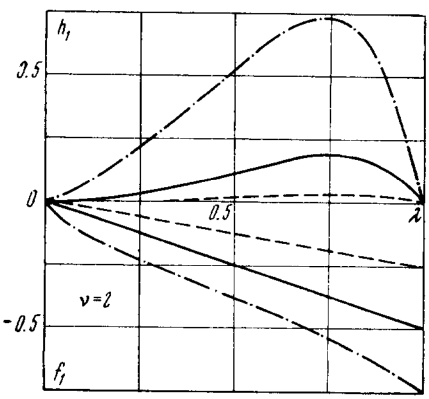

Fig. 2

Curves of h_1, f_1 and g_1 in Figs. 1 and 2 define the difference between the calculated pressure, velocity and density and their respective self-similar values for various laws of distribution of initial density and pressure for spherical (Fig. 1, $\nu = 3$) and cylindrical (Fig. 2, $\nu = 2$) symmetry for $\gamma = 3$, 7 and $5/3$; related values of ω are determined by formula (2.9). In Fig. 1 the solid lines relate to $\gamma=3$ and $\varkappa=0$, dash lines to $\gamma=7$ and $\varkappa={}^1/_3$, and the dot-dash lines to $\gamma=1.5$ and $\varkappa=2.4$. In Fig. 2 solid lines relate to $\gamma=3$ and $\varkappa=0$, dash lines to $\gamma=7$ and $\varkappa=-1$, and the dot-dash lines to $\gamma={}^5/_3$ and $\varkappa=1$. Curves of h_1, f_1 and g_1 calculated for constant initial density but varying initial

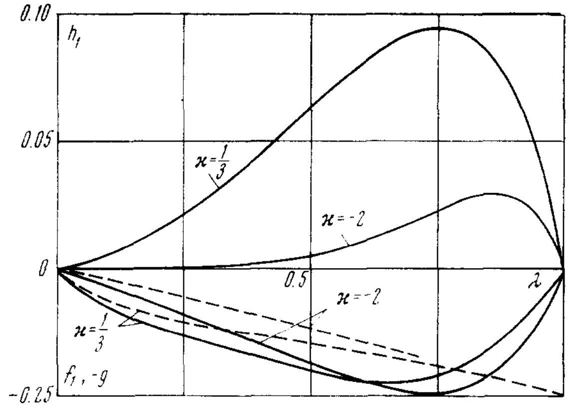

Fig. 3

pressure appear in Fig. 3 for $\nu=3$ and $\gamma=7$, with the f_1 curve shown by dash lines.

3. Solution of the nonlinearized problem. At the initial explosion stage, when the explosion wave is still fairly strong (for small q), the motion of gas is defined by a linearized solution which can be used for specifying initial conditions necessary for calculating the complete non-self-similar problem by the approximate analytical or numerical method [7, 10, 12].

Solution of the complete non-self-similar problem of explosion involves finding a solution of system (1.1) which would satisfy the boundary and initial conditions (1.2)–(1.5). The determination of functions $v_2(t)$, $\rho_2(t)$ and $p_2(t)$ defined by Eqs. (1.2) is tantamount to determining the shock wave radius $r_2(t)$. To do this, it is necessary to introduce a formula which would relate the explosion energy E_0 to parameters at the shock wave front. This relation can be expressed in terms of the energy conservation law, according to which the total energy of moving gas is, at every instant of time, equal to the sum of the initial energy of gas set in motion by the explosion and the energy E_0 of the explosion. We pass to dimensionless variables (1.8) in the law of total energy conservation and assume in what follows that the constant $\varkappa=\omega$ in the formula (0.1) of initial pressure distribution. We obtain

$$E_0+\delta_\nu\frac{p_1 r_2{}^\nu}{(\nu-\omega)(\gamma-1)}=\delta_\nu r_2{}^\nu\left[\frac{\rho_2 v_2{}^2}{2}\int_0^1\left(\frac{v}{v_2}\right)^2\left(\frac{\rho}{\rho_2}\right)\lambda^{\nu-1}\,d\lambda+\frac{p_2}{\gamma-1}\int_0^1\frac{p}{p_2}\lambda^{\nu-1}\,d\lambda\right]$$

$$\delta_\nu=2\pi(\nu-1)+(\nu-2)(\nu-3) \qquad (3.1)$$

The law of total energy conservation is convenient for determining the shock wave radius $r_2(t)$, when the solution of the system of Eqs. (1.1) has been found. System (1.1) with conditions (1.2)–(1.5) has a solution with all unknown functions of gasdynamic parameter distribution in explicit form, provided the dependence of the Euler coordinate r on t and on the Lagrangian coordinate ξ is known.

Let us seek r in the form

$$r = c(t)\,\xi^{\alpha_1(t)} + b \tag{3.2}$$

As the Lagrangian coordinate we take the initial coordinate of particle ξ. Since at the instant of the shock wave passing through the particle the Lagrangian coordinate $\xi = r_2$, the coefficients in (3.2) are

$$b = 0, \qquad c(t) = r_2^{1-\alpha_1(t)} \tag{3.3}$$

Using (3.2) and (3.3) from the law of mass conservation expressed in the differential form, we obtain

$$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{\alpha(t)} \tag{3.4}$$

$$\alpha(t) = \nu\left(\frac{\rho_2}{\rho_1} - 1\right) - \omega\frac{\rho_2}{\rho_1}, \qquad \alpha_1(t) = \frac{\rho_1}{\rho_2} \tag{3.5}$$

In the following we assume that the density distribution within the shock wave is determined by formulas (3.4) and (3.5). From the second equation of system (1.1) we can now determine the velocity and from the first, the pressure of gas. The third equation of system (1.1), the equation of energy, may be used for determining the shock wave radius $r_2(t)$. However the law of total energy conservation (3.1) is more convenient for determining this radius throughout the region of perturbed motion within the shock wave. We substitute the expression (3.4) for $\rho(r, t)$ into the second equation of system (1.1) and, taking into account the boundary condition (1.4) at the center, solve it for the velocity. We obtain

$$v(r, t) = v_2(t)\frac{r}{r_2} - \frac{r}{\alpha(t) + \nu}\frac{d\alpha}{dt}\ln\frac{r}{r_2} \tag{3.6}$$

We substitute the expressions (3.4) and (3.6) for $\rho(r, t)$ and $v(r, t)$, respectively, into the first of Eqs.(1.1) and solve it for the pressure, taking into account the boundary condition at the shock wave. We obtain

$$p(r, t) = p_2 + \frac{\rho_2 r_2 H_1}{\alpha(t) + 2} - \frac{\rho_2 r_2}{\alpha(t) + 2}\left(\frac{r}{r_2}\right)^{\alpha(t)}\left[H_1 + H_2\ln\frac{r}{r_2} + H_3\left(\ln\frac{r}{r_2}\right)^2\right] \tag{3.7}$$

where

$$H_1 = \frac{v_2^2}{r_2} + \frac{dv_2}{dt} - \frac{v_2}{r_2}\frac{dr_2}{dt} + \frac{dr_2}{dt}\frac{d}{dt}\left(\ln\frac{\rho_2}{\rho_1}\right) - v_2\frac{d}{dt}\left(\ln\frac{\rho_2}{\rho_1}\right) - \frac{H_2}{\alpha(t) + 2}$$

$$H_2 = 2H_3 - 2v_2\frac{d}{dt}\left(\ln\frac{\rho_2}{\rho_1}\right) - \frac{r_2\rho_1}{\rho_2}\frac{d^2}{dt^2}\left(\frac{\rho_2}{\rho_1}\right) - \frac{2H_3}{\alpha(t) + 2}$$

$$H_3 = r_2\left[\frac{d}{dt}\left(\ln\frac{\rho_2}{\rho_1}\right)\right]^2$$

Formulas (3.4), (3.6) and (3.7) which define the distribution of dimensionless parameters of motion in the perturbed region can, after some transformation, be written as

$$\frac{p}{p_2} = 1 + \frac{H_1\rho_2 r_2}{(\alpha + 2)\,p_2}\left(1 - \lambda^{\alpha+2}\right) - \frac{\rho_2 r_2}{p_2}\lambda^{\alpha+2}\left(H_2 + H_3\ln\lambda\right)\ln\lambda \tag{3.8}$$

$$\frac{v}{v_2} = (1 - H_4\ln\lambda)\,\lambda, \qquad \frac{\rho}{\rho_2} = \lambda^{\alpha}, \qquad \alpha = \nu\left(\frac{\rho_2}{\rho_1} - 1\right) - \omega\frac{\rho_2}{\rho_1}, \qquad \lambda = \frac{r}{r_2}$$

where

$$H_1 = -\frac{a^2}{r_0}\frac{2}{(\gamma + 1)\,q^2}\left[\frac{(1 - q^2)\,\psi}{(\gamma + 1)\,R_2} + \frac{(1 + 3q^2)}{q}\frac{dq}{dR_2}\right] - \frac{\psi}{\Omega - \nu\psi}H_2$$

$$H_2 = \frac{16a^2}{r_0}\left\{\frac{(1 - q^2)}{(\gamma + 1)\,\psi q}\frac{dq}{dR_2} - \frac{2R}{(\Omega - \nu\psi)\,\psi}\left(\frac{dq}{dR_2}\right)^2 + \frac{R}{4\psi q}\frac{d^2q}{dR_2^2}\right\}$$

$$H_3 = \frac{16a^2}{r_0}\frac{R}{\psi^2}\left(\frac{dq}{dR_2}\right)^2, \qquad H_4 = -\frac{2(\gamma + 1)\,Rq}{(1 - q^2)\,\psi}\frac{dq}{dR_2}$$

$$q = \frac{u}{c}, \quad \psi = \gamma - 1 + 2q^2, \quad \Omega = 2(\nu - \omega) + (\nu + 2 - \omega)(\gamma - 1) + \nu q^2$$

Solution (3. 8) satisfies all boundary conditions and is expressed in terms of parameters at the shock wave front and the front coordinate $r_2\ (t)$, The time t does not explicitly appear in this solution. It is evident from formulas (3. 8) and (1. 2) that all characteristics of motion of gas in the perturbed region can be expressed in terms of function $r_2\ (t)$ Subjecting solution (3. 8) to the law of total energy conservation (3. 1), we obtain the equation which can be used for determining the law of shock wave motion

$$R_2{}^2 q \frac{d^2q}{dR_2{}^2} - 4R_2{}^2q^2 \frac{[2 + \nu(\gamma - 1)]}{\Omega}\left(\frac{dq}{dR_2}\right)^2 + \frac{(1+3q^2)}{2(\gamma+1)}\frac{\Omega R_2}{q}\frac{dq}{dR_2} +$$
$$+ \frac{2(1-q^2)R_2q}{\gamma+1}[\nu(\gamma-1)+2]\frac{dq}{dR_2} + \frac{(1-q^2)\Omega}{2(\gamma+1)^2}[\psi - \Omega - \nu(\gamma-1)(1-q^2)] +$$
$$+ \frac{\nu(\gamma-1)\Omega^2q^2}{4\gamma(\gamma+1)\delta_\nu R_2^{\nu-\omega}} - \frac{\omega\Omega^2q^2}{(\nu-\omega)4\gamma(\gamma+1)} = 0 \qquad (3.9)$$

For the numerical integration of Eq.(3. 9) we reduce it to a system of two first order equations. Taking the parameter $q = u / c$ as the independent variable and $R_2\ (q)$ and

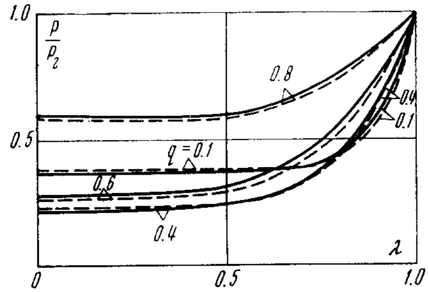

Fig. 4

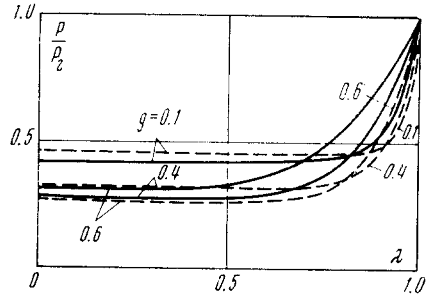

Fig. 5

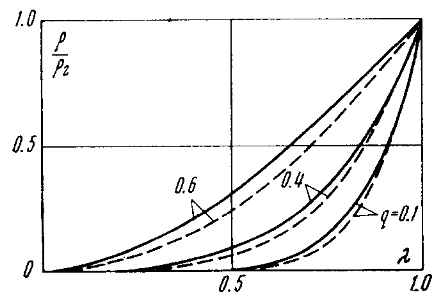

Fig. 6

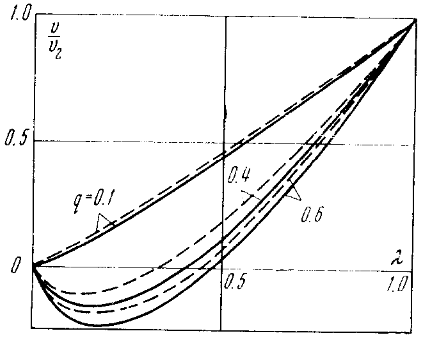

Fig. 7

$\sigma\ (q) = dq\ /\ dR_2$, as the unknown functions, after some transformations we obtain

$$\frac{d\sigma}{dq} = \frac{4[2+\nu(\gamma-1)]q}{\Omega}\sigma - \frac{(1+3q^2)}{2(\gamma+1)R_2q^2}\Omega - \frac{2(1-q^2)[2+\nu(\gamma-1)]}{(\gamma+1)R_2} - \qquad (3.10)$$
$$- \frac{(1-q^2)\Omega}{2(\gamma+1)^2R_2{}^2q\sigma}[\psi - \Omega - \nu(\gamma-1)(1-q^2)] -$$

$$\frac{\nu(\gamma-1)\Omega^2 q}{4\gamma(\gamma+1)\sigma\delta_\nu R^{\nu+2-\omega}}+\frac{\omega\Omega^2 q}{4(\nu-\omega)(\gamma+1)\sigma R_2^2}$$

$$\frac{dR_2}{dq}=\frac{1}{\sigma}$$

The initial condition for the solution of system (3.10) is to be defined at point $q=0$, $R_2=0$. This is a singular point in the neighborhood of which the asymptotic behavior is defined by

$$q=C_0 R_2^{\nu-\omega/2},\qquad C_0^2=\frac{\delta_\nu\gamma\,[2(\gamma-1)(\gamma-3)-3(\nu-\omega)(\gamma+1)]}{\nu(\gamma^2-1)\,[2(\nu-\omega)+(\nu+2-\omega)(\gamma-1)]}\tag{3.11}$$

The system of Eqs.(3.10) was integrated by the Runge-Kutta method for the asymptotics (3.11) for spherical symmetry ($\nu=3$) and several values of the adiabatic exponent γ and exponent ω in the laws of density and pressure distribution.

The distribution of dimensionless pressure fields for $\gamma={}^5/_3$ and $\gamma=1.4$ are shown, respectively, in Figs. 4 and 5 for values of parameters q and ω (solid lines relate to $\omega=0.5$ and dash lines to $\omega=0.33$). The distribution of dimensionless density is shown in Fig. 6 for $\gamma={}^5/_3$ and several values of q and ω ($\omega=0.5$; 0.33). Dimensionless velocities for $\omega=0.5$ and several values of q and γ are shown in Fig. 7, where solid lines relate to $\gamma=1.4$, and the dash lines to $\gamma={}^5/_3$). Figures 4 and 5 show that for a given γ the variation of dimensionless pressure with decreasing ω is nonmonotonic, while it can be seen from Fig. 6 that the dimensionless density decreases with decreasing ω. Figure 7 shows that for a specified $\omega=0.5$ dimensionless velocities increase with increasing γ.

BIBLIOGRAPHY

1. Sedov, L. I., Methods of Similarity and Dimensionality in Mechanics. Gostekhizdat, Moscow, 1957.
2. Sedov, L. I., Motion of air in a strong explosion. Dokl. Akad. Nauk SSSR, Vol. 52, №1, 1946.
3. Korobeinikov, V. P., Mel'nikova, N. S. and Riazanov, E. V., Theory of Point Explosion. Fizmatgiz, Moscow, 1961.
4. Okhotsimskii, D. E., Kondrashova, I. L., Vlasova, Z. P. and Kazakova, R. K., Calculation of point explosion with allowance for counterpressure. Tr. Matem. Inst. Akad. Nauk SSSR, Vol. 50, 1957.
5. Sakurai, A., On propagation and structure of the blast wave. J. Phys. Soc. Japan, Vol. 8, №5, 1953 and Vol. 9, №2, 1954.
6. Mel'nikova, N. S. and Salamakhin, T. M., On the calculation of point explosion in various gases. PMTF, №4, 1964.
7. Korobeinikov, V. P. and Chushkin, P. I., Methods of calculation of point explosions in gases. Dokl. Akad. Nauk SSSR, Vol. 154, №3, 1964.
8. Korobeinikov, V. P. and Mel'nikova, N. S., On exact solutions of the linearized problem of point explosion with counterpressure. Dokl. Akad. Nauk SSSR, Vol. 116, №2, 1957.
9. Korobeinikov, V. P. and Riazanov, E. V., On the theory of linearized problems on explosion taking counterpressure into account. PMM Vol. 23, №4, 1959.
10. Mel'nikova, N. S., On the point explosion in a medium of varying initial density. Tr. Matem. Inst. Akad. Nauk SSSR, Vol. 37, 1966.

11. Karlikov, V.P., Solution of the linearized axisymmetric problem of point explosion in a medium of varying density. Dokl. Akad. Nauk SSSR, Vol. 101, №6, 1955.
12. Korobeinikov, V.P., On the gas flow due to solar flares. J. Solar Res. and Study, Solar Terrestrial Phys. №7, 1969.
13. Korobeinikov, V.P., On the application of dimensional analysis in problems of interplanetary gas motion in solar flares. Dokl. Akad. Nauk SSSR, Vol. 185, №6, 1969.

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WAVES GENERATED BY PERTURBATIONS OF THE BOTTOM OF A TANK WITH A DOCK

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A wave motion generated on the surface of a heavy incompressible fluid by oscillations of a section of the bottom of a tank with a dock is studied. The problem of waves generated by an oscillating section of the bottom of a tank was dealt with in [1, 2]. In the present paper the Wiener-Hopf method [3] is employed to solve the analogous problem in which the boundary conditions have been altered, namely, a part of the free surface is covered with an immovable rigid plate. An expression for the velocity potential describing the motion of the fluid in the problem under consideration is derived. The results of [2, 4, 5] are found to be particular cases of the solution obtained here. The numerical example given shows that the rise of the free surface is smaller on the dock side than that at the corresponding point at the side opposite to the oscillating section of the bottom.

1. An immovable rigid plate is situated at the surface of a fluid of finite depth h, occupying the region $y = h$, $x \leqslant -l$ and $-\infty < z < \infty$. The coordinate origin is placed at the bottom of the tank and the y-axis is directed vertically upwards. The section $y = 0$, $0 \leqslant x \leqslant a$, $-\infty < z < \infty$ of the bottom undergoes vertical displacement according to the law

$$y = \operatorname{Re}[v(x) \exp i(kz - \omega t)]$$

where $v(x)$ is a numerically small, smooth function. The velocity potential $F(x, y, z, t)$ which in this case describes the motion of the fluid, must satisfy the following boundary value problem

$$\Delta F(x, y, z, t) = 0 \quad (0 \leqslant y \leqslant h, -\infty < x < \infty, -\infty < z < \infty)$$

$$\partial^2 F / \partial t^2 + g \partial F / \partial y = 0 \text{ when } y = h, \quad x > -l, \quad -\infty < z < \infty$$

$$\partial F / \partial y = 0 \text{ when } y = h, \quad x \leqslant -l, \quad -\infty < z < \infty \tag{1.1}$$

$$\partial F / \partial y = \begin{cases} -\omega \operatorname{Re}[iv(x) \exp i(kz - \omega t)] & (0 \leqslant x \leqslant a) \\ 0 & (-\infty < x < 0, a < x < \infty) \end{cases} \quad y = 0, \quad -\infty < z < \infty$$